Chapter 1
Introduction

1.1 Background

It took a lot of years to human perception to assimilate the following notation :

$$2 + 2 + 2 = 2 \cdot 3$$
$$2 \cdot 2 \cdot 2 = 2 \wedge 3$$

It seems worth therefore to continue exploring the benefits of this notation abstraction one order higher. We define this way

$$2 \wedge 2 \wedge 2 = 2 \otimes 3$$

Here we used the symbol $\otimes$ to denote what we called hypher-exponentiation.

Now, consider the equation $a \odot b = c$, where $\odot$ one of $\{+, \cdot, \wedge, \otimes\}$ and let one of the a,b or c be unknown. We take the following cases :

1) $a + b = c$

$$\begin{cases} a + b = x \rightarrow x = a + b & \text{(addition)} \\ a + x = c \rightarrow x = c - a & \text{(substraction)} \\ x + b = c \rightarrow x = c - b & \text{(substraction)} \end{cases}$$

2) $a \cdot b = c$

$$\begin{cases} a \cdot b = x \rightarrow x = a \cdot b & \text{(multiplication)} \\ a \cdot x = c \rightarrow x = \frac{c}{a} & \text{(division)} \\ x \cdot b = c \rightarrow x = \frac{c}{b} & \text{(division)} \end{cases}$$

3) $a \wedge b = c$

$$\begin{cases} a \wedge b = x \rightarrow x = a \wedge b & \text{(power)} \\ a \wedge x = c \rightarrow x = \log_a c & \text{(logarithm)} \\ x \wedge b = c \rightarrow x = \sqrt[b]{c} & \text{(root)} \end{cases}$$

4) $a \otimes b = c$

$$\begin{cases} a \otimes b = x \to x = a \otimes b & \text{(hypherpower)} \\ a \otimes x = c \to x = \mathrm{HLog}_a\, c & \text{(hypher-logarithm)} \\ x \otimes b = c \to x = \mathrm{HRoot}_a\, c & \text{(hypher-root)} \end{cases}$$

Having in mind these basic structures, let us examine more closely hypher-powers. Observe that the notation

$$2 \wedge 2 \wedge 2 \wedge 2 = 2 \otimes 4$$

may be given two different interpretations since exponentiation is not commutative. That is we may interpret the notation with one of the following ways

$$((2 \wedge 2) \wedge 2) \wedge 2 = 2 \otimes 4 = 256 \quad \text{or}$$
$$2 \wedge (2 \wedge (2 \wedge 2)) = 2 \otimes 4 = 65536$$

In the following we shall turn our attention to the first interpretation. Also we will use the notation

$$\binom{4}{\hat{2}} = \left(\left(2^2\right)^2\right)^2$$

to denote "2 to the hypher-power 4" instead of $2 \otimes 4$, which we used just for the purposes of the current section. Thus, we adopt that

$$\binom{4}{\hat{2}} = \left(\left(2^2\right)^2\right)^2 = \left(2^2\right)^{2 \cdot 2} = 2^{2 \cdot 2 \cdot 2}$$

and in general, we shall define

$$\binom{n}{\hat{a}} = a^{\overbrace{a \cdot a \cdots a}^{n-1 \text{ times}}} = a^{a^{n-1}}$$

Observe that the definition makes sense also when n takes real values.

1.2 Motivation

The backbone of our investigations will be to transfer known formulae or identities one order higher in our symbolic hierarchy and check the laws that hold under this mapping. In summary, we will use the operation mapping as appears in the following table

$a + b = x$	$+ \to \cdot$	$- \to \div$	$- \to \div$
$a \cdot b = x$	$\cdot \to \wedge$	$\div \to \log$	$\div \to \sqrt{\ }$
$a \wedge b = x$	$\wedge \to \otimes$	$\log \to \mathrm{HLog}$	$\sqrt{\ } \to \mathrm{HRoot}$

1.3 Properties

Concerning addition and multiplication of hypher-powers (low order operators), results to more complex expressions. Lets try raising to a power with the same base :

$$\binom{n}{\hat{a}}^{(a^m)} = \left(a^{a^{n-1}}\right)^{(a^m)}$$
$$= a^{a^{n-1}a^m} = a^{a^{n+m-1}} = \binom{n+m}{\hat{a}}$$

Thus we conclude

$$\binom{n}{\hat{a}}^{(a^m)} = \binom{n+m}{\hat{a}}$$

1.4 An exponential sequence

Consider the following sequence where $\log_{a_n} a_{n+1} = k$

$$a_0$$
$$a_1 = a_0^k$$
$$a_2 = a_1^k = (a_0^k)^k = a_0^{k^2}$$
$$a_3 = a_2^k = ((a_0^k)^k)^k = a_0^{k^3}$$

Then

$$P_n = a_0 \cdot a_0^k \cdot a_0^{k^2} \cdot a_0^{k^3} \cdots a_0^{k^n}$$
$$P_n^k = a_0^k \cdot a_0^{k^2} \cdot a_0^{k^3} \cdot a_0^{k^4} \cdots a_0^{k^{n+1}}$$

so that

$$\frac{P_n^k}{P_n} = \frac{a_0^{k^{n+1}}}{a_0}$$
$$P_n = a_0^{\frac{k^{n+1}-1}{k-1}}$$

1.5 Derivative of a hypher-power

In order to get accustomed with hypher-powers and understand their nature, lets calculate some derivatives.

$$\begin{aligned}\frac{d}{dx}\binom{x+1}{\hat{2}} &= \frac{d}{dx}2^{2^x}\\ &= \ln(2)\,2^{2^x}\frac{d}{dx}2^x\\ &= \ln(2)\ln(2)\,2^{2^x}2^x\end{aligned}$$

also

$$\begin{aligned}\frac{d}{dx}\binom{3}{\hat{x}} &= \frac{d}{dx}x^{x^2}\\ &= x^{x^2}\frac{d}{dx}(\ln(x)\cdot x^2)\\ &= x^{x^2}\left(\frac{x^2}{x}+\ln(x)\cdot 2x\right)\\ &= x^{x^2}\Big(x+2x\cdot\ln(x)\Big)\end{aligned}$$

and one more

$$\begin{aligned}\frac{d}{dx}\binom{4}{\hat{x}} &= \frac{d}{dx}x^{x^3}\\ &= x^{x^3}\frac{d}{dx}(\ln(x)\cdot x^3)\\ &= x^{x^3}\left(\frac{x^3}{x}+\ln(x)\cdot 3x^2\right)\\ &= x^{x^3}\Big(x^2+3x^2\cdot\ln(x)\Big)\end{aligned}$$

Chapter 2
Hypher-Differentiation

2.1 Hypher-differentiation

Let us consider ordinary differentiation, as defined by the type

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

and translate the above formula one order higher. We get

$$\lim_{h\to 1} \log_h \frac{f(x\cdot h)}{f(x)}$$

the limit above (should be proved) can be written equivalently as below

$$\lim_{h\to 1} \log_h \frac{f(x\cdot h)}{f(x)} \Leftrightarrow x\cdot \lim_{h\to 0} \log_{1+h} \frac{f(x+h)}{f(x)}$$

2.2 Hypher-derivatives of common functions

Hypher-derivative of an ordinary power

$$
\begin{aligned}
(x^2)^{\triangledown} &= \lim_{h\to 1} \log_h \frac{(xh)^2}{x^2} \\
&= \lim_{h\to 1} \log_h h^2 \\
&= \lim_{h\to 1} \frac{\log h^2}{\log h} \\
&= \lim_{h\to 1} \frac{\frac{d}{dh}\log h^2}{\frac{d}{dh}\log h} \\
&= \lim_{h\to 1} \frac{\frac{2}{h}}{\frac{1}{h}} \\
&= \lim_{h\to 1} 2 = 2
\end{aligned}
$$

Hypher-derivative of an ordinary exponential

$$
\begin{aligned}
(2^x)^{\triangledown} &= \lim_{h\to 1} \log_h \frac{2^{(xh)}}{2^x} \\
&= \lim_{h\to 1} \log_h 2^{xh-x} \\
&= \lim_{h\to 1} \frac{\log 2^{xh-x}}{\log h} \\
&= \lim_{h\to 1} \frac{\frac{d}{dh}\log 2^{xh-x}}{\frac{d}{dh}\log h} \\
&= \lim_{h\to 1} \frac{2^{xh-x} \cdot x \cdot \log(2)}{\frac{1}{h}} \\
&= x\log(2)
\end{aligned}
$$

Hypher-derivative of a hypher-power 1st form

Let us calculate

$$\binom{3}{\hat{x}}^{\nabla}$$

$$= \lim_{h\to 1} \log_h \frac{(xh)^{(xh)^2}}{x^{x^2}} = \lim_{h\to 1} \log_h \frac{x^{(xh)^2} h^{(xh)^2}}{x^{x^2}}$$

$$= \lim_{h\to 1} \log_h x^{x^2(h^2-1)} + \lim_{h\to 1} \log_h h^{x^2h^2}$$

$$= \lim_{h\to 1} x^2(h^2-1)\log_h x + \lim_{h\to 1} x^2h^2\log_h h$$

$$= \log x \lim_{h\to 1} \frac{x^2(h^2-1)}{\log h} + x^2 \lim_{h\to 1} h^2 \log_h h$$

$$= \log x \lim_{h\to 1} \frac{\frac{d}{dh}x^2(h^2-1)}{\frac{d}{dh}\log h} + x^2$$

$$= \log x \lim_{h\to 1} \frac{2hx^2}{\frac{1}{h}} + x^2$$

$$= \log x \cdot 2x^2 + x^2 = x^2(2\log x + 1) = x^2 \log(ex^2)$$

Hypher-derivative of a hypher-power 2nd form

Let us calculate

$$\binom{x}{\hat{2}}^{\nabla}$$

$$= \lim_{h\to 1} \log_h \frac{2^{2^{(xh-1)}}}{2^{2^{x-1}}} = \dots$$

$$= \log^2(2)\, x 2^{x-1}$$

so that holds

$$\binom{x}{\hat{e}}^{\nabla} = x \cdot e^{x-1}$$

and in general (to be checked)

$$\binom{f(x)}{\hat{e}}^{\nabla} = x \cdot f'(x) \cdot e^{f(x)-1}$$

which should be compared with

$$\left(e^{f(x)}\right)' = f'(x) \cdot e^{f(x)}$$

2.3 Hypher-derivatives of trigonometric functions

Hypher-derivative of sin x

$$(\sin x)^{\nabla} = \lim_{h\to 1} \log_h\left(\frac{\sin xh}{\sin x}\right) = \lim_{h\to 1} \frac{\log\left(\frac{\sin xh}{\sin x}\right)}{\log h}$$

$$= \lim_{h\to 1} \frac{\frac{d}{dh}\log\left(\frac{\sin xh}{\sin x}\right)}{\frac{d}{dh}\log h} = \lim_{h\to 1} \frac{\frac{\sin x}{\sin xh}\,\frac{d}{dh}\left(\frac{\sin xh}{\sin x}\right)}{\frac{1}{h}}$$

$$= \lim_{h\to 1} h\frac{1}{\sin xh}\,\frac{d}{dh}(\sin xh) = \lim_{h\to 1} xh\frac{\cos xh}{\sin xh} = \lim_{h\to 1} xh\,(\cot xh)$$

$$= x\ \cot x$$

Hypher-derivative of cos x

$$(\cos x)^{\nabla} = \lim_{h\to 1} \log_h\left(\frac{\cos xh}{\cos x}\right) = \lim_{h\to 1} \frac{\log\left(\frac{\cos xh}{\cos x}\right)}{\log h}$$

$$= \lim_{h\to 1} \frac{\frac{d}{dh}\log\left(\frac{\cos xh}{\cos x}\right)}{\frac{d}{dh}\log h} = \lim_{h\to 1} \frac{\frac{\cos x}{\cos xh}\,\frac{d}{dh}\left(\frac{\cos xh}{\cos x}\right)}{\frac{1}{h}}$$

$$= \lim_{h\to 1} h\frac{1}{\cos xh}\,\frac{d}{dh}(\cos xh) = \lim_{h\to 1}(-xh)\frac{\sin xh}{\cos xh} = \lim_{h\to 1}(-xh)\,(\tan xh)$$

$$= -x\ \tan x$$

Hypher-derivative of tan x

$$(\tan x)^{\nabla} = \lim_{h\to 1} \log_h\left(\frac{\tan xh}{\tan x}\right) = \lim_{h\to 1} \frac{\log\left(\frac{\tan xh}{\tan x}\right)}{\log h}$$

$$= \lim_{h\to 1} \frac{\frac{d}{dh}\log\left(\frac{\tan xh}{\tan x}\right)}{\frac{d}{dh}\log h} = \lim_{h\to 1} \frac{\frac{\tan x}{\tan xh}\,\frac{d}{dh}\left(\frac{\tan xh}{\tan x}\right)}{\frac{1}{h}}$$

$$= \lim_{h\to 1} h\frac{1}{\tan xh}\,\frac{d}{dh}(\tan xh) = \lim_{h\to 1}(xh)\frac{\sec^2 xh}{\tan xh}$$

$$= \lim_{h\to 1}(xh)\,(\csc\ xh)\,(\sec\ xh) = x\ \csc x\ \sec x$$

Continuing this way, we get the hypher-directives of the other trigonometric functions as well. The table following sumarizes our results

$$(\sin x)^{\nabla} = x \cot x$$

$$(\cos x)^{\nabla} = -x \tan x$$

$$(\tan x)^{\nabla} = x \csc x \sec x$$

$$(\cot x)^{\nabla} = \left(\frac{1}{\tan x}\right)^{\nabla} = -x \csc x \sec x$$

$$(\sec x)^{\nabla} = \left(\frac{1}{\cos x}\right)^{\nabla} = x \tan x$$

$$(\csc x)^{\nabla} = \left(\frac{1}{\sin x}\right)^{\nabla} = -x \cot x$$

2.4 Hypher-derivative of a polynomial

$$(1+x)^{\nabla} = \frac{x}{x+1}$$

$$(1+x+x^2)^{\nabla} = \frac{x(1+2x)}{1+x+x^2}$$

$$(1+x+x^2+x^3)^{\nabla} = \frac{x(1+2x+3x^2)}{1+x+x^2+x^3}$$

$$(1+3x+5x^2)^{\nabla} = \frac{x(3+10x)}{1+3x+5x^2}$$

therefore to be proved

$$P^{\nabla}(x) = \frac{xP'(x)}{P(x)}$$

$$\begin{aligned}
P^{\nabla}(x) &= \lim_{h\to 1} \log_h \frac{P(xh)}{P(x)} = \lim_{h\to 1} \frac{\log \frac{P(xh)}{P(x)}}{\log h} \\
&= \lim_{h\to 1} \frac{\frac{d}{dh} \log \frac{P(xh)}{P(x)}}{\frac{d}{dh} \log h} = \lim_{h\to 1} \frac{\frac{P(x)}{P(xh)} \frac{d}{dh} \frac{P(xh)}{P(x)}}{\frac{1}{h}} \\
&= \lim_{h\to 1} h \frac{1}{P(xh)} \frac{d}{dh} P(xh) = \lim_{h\to 1} xh \frac{1}{P(xh)} \frac{d}{dx} P(x) \\
&= x \frac{1}{P(x)} P'(x) = \frac{xP'(x)}{P(x)}
\end{aligned}$$

also holds

$$\left(\frac{1}{P(x)}\right)^{\nabla} = -P^{\nabla}(x)$$

$$\left(P^{-1}(x)\right)^{\nabla} = -P^{\nabla}(x)$$

we would like also to check

$$P^{\nabla\nabla}(x) = 1 - P^{\nabla}(x) + \frac{xP''(x)}{P'(x)}$$

$$f^{\nabla}(x) = x(\log f(x))'$$

continuing the same way we may write

$$f^{\nabla} = x(\log f)'$$

$$f^{\nabla\nabla} = x(\log f^{\nabla})'$$

$$f^{\nabla\nabla\nabla} = x(\log f^{\nabla\nabla})'$$

$$f^{\nabla\nabla\nabla\nabla} = x(\log f^{\nabla\nabla\nabla})'$$

we may rewrite these as follows

$$f = e^{\int \frac{f^{\nabla}}{x}dx}$$

$$f^{\nabla} = e^{\int \frac{f^{\nabla\nabla}}{x}dx}$$

$$f^{\nabla\nabla} = e^{\int \frac{f^{\nabla\nabla\nabla}}{x}dx}$$

$$f^{\nabla\nabla\nabla} = e^{\int \frac{f^{\nabla\nabla\nabla\nabla}}{x}dx}$$

2.5 Useful identities

$$\frac{f^{\nabla}}{x} = \frac{f'}{f}$$

$$(\log f)^{\nabla} = \frac{f^{\nabla}}{\log f}$$

$$f = e^{\frac{f^{\nabla}}{\log f^{\nabla}}}$$

$$\begin{aligned}\left(h^{h-1-\log h}\right)^{\frac{1}{\log h}} &= h^{\frac{h-1-\log h}{\log h}} \\ &= e^{\log h^{\frac{h-1-\log h}{\log h}}} \\ &= e^{\frac{h-1-\log h}{\log h}\cdot \log h} \\ &= e^{h-1-\log h} \\ &= \frac{e^h}{e\cdot h}\end{aligned}$$

$$(e^f)^{\nabla} = f\cdot f^{\nabla}$$

$$(f^n)^{\nabla} = n\cdot f$$

2.6 Hypher-differentiation rules

Product rule The hypher-derivative of the product p of two functions f and g, equals the sum of their hypher-derivatives.

$$
\begin{aligned}
p^{\nabla}(x) &= (f \cdot g)^{\nabla}(x) \\
&= \lim_{h \to 1} \log_h \frac{f(xh) \cdot g(xh)}{f(x) \cdot g(x)} \\
&= \lim_{h \to 1} \log_h \frac{f(xh)}{f(x)} + \lim_{h \to 1} \log_h \frac{g(xh)}{g(x)} \\
&= f^{\nabla}(x) + g^{\nabla}(x)
\end{aligned}
$$

Chain rule The hypher-derivative of the composition of two functions f and g, is computed according to the following rule:

$$(f \circ g)^{\nabla}(x) = f^{\nabla}(g(x)) \cdot g^{\nabla}(x)$$

Example 1:

$$f(x) = 3x + 1 \qquad \to f^{\nabla} = \frac{3x}{3x+1}$$

$$g(x) = x^2 \qquad \to g^{\nabla} = 2$$

$$(f \circ g)^{\nabla}(x) = (3x^2 + 1)^{\nabla}(x) = x\frac{(3x^2+1)'}{3x^2+1} = \frac{6x^2}{3x^2+1}$$

$$f^{\nabla}(g(x)) \cdot g^{\nabla}(x) = \frac{3x^2}{3x^2+1} \cdot 2 = \frac{6x^2}{3x^2+1}$$

Example 2:

$$f(x) = \sin x \qquad \to f^{\nabla} = x\cot x$$

$$g(x) = x^2 \qquad \to g^{\nabla} = 2$$

$$(g \circ f)^{\nabla}(x) = (\sin^2(x))^{\nabla} = x\frac{(\sin^2(x))'}{\sin^2(x)} = 2x\cot x$$

$$g^{\nabla}(f(x)) \cdot f^{\nabla}(x) = 2 \cdot x\cot x$$

Sum rule The hypher-derivative of the sum s of two functions f and g, equals with the weighted sum of their hypher-derivatives.

$$\begin{aligned} s^{\nabla}(x) &= (f+g)^{\nabla}(x) \\ &= \frac{x(f+g)'}{f+g} = \frac{xf'+xg'}{f+g} \\ &= \frac{xf'f}{f(f+g)} + \frac{xg'g}{g(f+g)} \\ &= f^{\nabla}\frac{f}{f+g} + g^{\nabla}\frac{f}{f+g} \end{aligned}$$

Example 1:

$$(x^3+1)^{\nabla} = (x^3)^{\nabla}\frac{x^3}{x^3+1} + 1^{\nabla}\frac{x^3}{x^3+1} = \frac{3x^3}{x^3+1}$$

Lemma

$$\begin{aligned} (f_1+f_2+\cdots f_n)^{\nabla} &= \frac{f_1\cdot f_1^{\nabla} + f_2\cdot f_2^{\nabla} + \cdots f_n\cdot f_n^{\nabla})}{f_1+f_2+\cdots f_n} \\ &= \frac{(e_1^f)^{\nabla} + (e_2^f)^{\nabla} + \cdots (e_n^f)^{\nabla}}{f_1+f_2+\cdots f_n} \end{aligned}$$

Power rule The hypher-derivative of the power of a functions f to the g, may be computed as follows:

$$\begin{aligned} (f^g)^{\nabla} &= \\ &= \frac{x(f^g)'}{f^g} \\ &= \frac{x(e^{g\log f})'}{f^g} \\ &= \frac{xf^g(g\log f)'}{f^g} \\ &= x(g\log f)' \\ &= x(g\frac{f\prime}{f} + g\prime\log f) \\ &= f^{\nabla}g + g^{\nabla}\log(f^g) \end{aligned}$$

Example 1:

$$\left((e^x)^{x^2}\right)^{\nabla} = x(x^2\log e^x)' = x(x^3)' = 3x^3$$

2.7 Determining function from its logarithmic derivatives

The question is : If we know the logatithmic derivatives of a function, is then the function determined?

2.8 Constant 1st order logatithmic derivative

We consider the class of functions with general form

$$f(x) = a \cdot x^b$$

Example 1: Let $f^{\triangledown}(x) = 2$, then

$$f(x) = e^{\int \frac{2}{x} dx} = e^{2\ln|x|+c} = a \cdot x^2$$
$$f(x \cdot h) = f(x) \cdot h^{f^{\triangledown}(x)} = (a \cdot x^2) \cdot h^2 = a \cdot (x \cdot h)^2$$

$$f(x \cdot h) = f(x) \cdot h^{f^{\triangledown}(x)} \quad \text{is exact}$$

2.9 Constant 2nd order logatithmic derivative

We consider the class of functions with general form

$$f(x) = a \cdot A^{b \cdot x^c}$$

Let f be such that the 2nd logatithmic derivative is constant $f^{\nabla\nabla} = c$, then :

$$(1) \quad f(xh) = f(x) \cdot h^{f^{\nabla}(x)} \cdot c_2$$

Replacing f with f^{∇} and knowing that f^{∇} is exact, we have:

$$(2) \quad f^{\nabla}(xh) = f^{\nabla}(x) \cdot h^{f^{\nabla\nabla}(x)}$$

Differentiating (1) we get

$$f^{\nabla}(xh) = f^{\nabla}(x) + (h^{f^{\nabla}(x)})^{\nabla} + c_2^{\nabla}$$

$$(3) \quad f^{\nabla}(xh) = f^{\nabla}(x) + f^{\nabla}(x) \cdot f^{\nabla\nabla}(x) \log h + c_2^{\nabla}$$

from (2) and (3) and omitting x we get

$$f^{\nabla} + f^{\nabla} \cdot f^{\nabla\nabla} \log h + c_2^{\nabla} = f^{\nabla} \cdot h^{f^{\nabla\nabla}}$$

$$c_2^{\nabla} = f^{\nabla} \cdot h^{f^{\nabla\nabla}} - f^{\nabla} - f^{\nabla} \cdot f^{\nabla\nabla} \log h$$

$$c_2^{\nabla} = f^{\nabla} \cdot (h^{f^{\nabla\nabla}} - 1 - \log h^{f^{\nabla\nabla}})$$

$$c_2^{\nabla} = f^{\nabla} \cdot (h^c - 1 - \log h^c)$$

then

$$c_2 = e^{\int \frac{c_2^{\nabla}}{x} dx} = e^{\int \frac{f^{\nabla} \cdot (h^c - 1 - \log h^c)}{x} dx} = e^{(h^c - 1 - \log h^c) \cdot \int \frac{f^{\nabla}}{x} dx}$$

$$= e^{(h^c - 1 - \log h^c) \cdot (\log f + k)} = \left(\frac{e^{h^c}}{e \cdot h^c} \right)^{(\log f + k)}$$

Now, for $x = 0$ we have $f^{\nabla}(0) = 0$ and so $c_2 = 1$. Thus it should be $\log f(0) + k = 0$ and so $k = -\log a$. Also notice the following

$$\begin{cases} f = a \cdot A^{b \cdot x^c} \\ f^{\nabla} = c \cdot b \cdot x^c \log A \\ f^{\nabla\nabla} = c \\ \log f = \log a + b \cdot x^c \log A \end{cases} \rightarrow \log f - \log a = \frac{f^{\nabla}}{f^{\nabla\nabla}}$$

To conclude finally that

$$f(x \cdot h) = f(x) \cdot h^{f^{\nabla}(x)} \cdot \left(\frac{e^{h^{f^{\nabla\nabla}}}}{e \cdot h^{f^{\nabla\nabla}}} \right)^{\frac{f^{\nabla}}{f^{\nabla\nabla}}}$$

Example 1. Consider the function

$$f(x) = 2^x \text{ for which}$$
$$f^{\nabla} = x \log 2$$
$$f^{\nabla\nabla} = 1$$

Then for x = 5 and h = 0:1

1) $f(x \cdot h) = 2^{0:5} = \sqrt{2} = 1{,}414213562373095$

2) $f(x \cdot h) = 2^5 \cdot 0{,}1^{5 \log 2} \cdot \left(\frac{e^{0{,}1^1}}{e \cdot 0{,}1^1}\right)^{\frac{5 \log 2}{1}}$
$= 1{,}4142135623730951$

Example 2. Consider the function

$$f(x) = 4 \cdot 2^{3 \cdot x^2} \text{ for which}$$
$$f^{\nabla} = 6 \cdot x \log 2$$
$$f^{\nabla\nabla} = 2$$

Then for x = 2,5 and h = 0,2

1) $f(x \cdot h) = 4 \cdot 2^{3 \cdot (2{,}5 \cdot 0{,}2)^2} = 6{,}7271713220297163$

2) $f(x \cdot h) = 4 \cdot 2^{3 \cdot 2{,}5^2} \cdot 0{,}2^{6 \cdot 2{,}5^2 \log 2} \cdot \left(\frac{e^{0{,}2^2}}{e \cdot 0{,}2^2}\right)^{\frac{6 \cdot 2{,}5^2 \log 2}{2}}$
$= 6{,}7271713220297078$

2.10 Constant 3^{rd} order logatithmic derivative

Let f be such that the 3^{rd} logatithmic derivative is constant $f^{\nabla\nabla\nabla} = c$, then :

$$(1) \quad f(x \cdot h) = f(x) \cdot h^{f^{\nabla}(x)} \cdot \left(\frac{e^{h^{f^{\nabla\nabla}}}}{e \cdot h^{f^{\nabla\nabla}}}\right)^{\frac{f^{\nabla}}{f^{\nabla\nabla}}} \cdot c_3$$

Replacing f with f^{∇} and knowing that $f^{\nabla\nabla\nabla}$ is exact, we have:

$$(2) \quad f^{\nabla}(x \cdot h) = f^{\nabla}(x) \cdot h^{f^{\nabla\nabla}(x)} \cdot \left(\frac{e^{h^{c}}}{e \cdot h^{c}}\right)^{\frac{f^{\nabla\nabla}}{c}}$$

Differentiating (1) we get

$$(3) \quad f^{\nabla}(x \cdot h) = f^{\nabla}(x) + \left(h^{f^{\nabla}(x)}\right)^{\nabla} + \left(\left(\frac{e^{h^{f^{\nabla\nabla}}}}{e \cdot h^{f^{\nabla\nabla}}}\right)^{\frac{f^{\nabla}}{f^{\nabla\nabla}}}\right)^{\nabla} + c_3^{\nabla}$$

$$= f^{\nabla} + f^{\nabla\nabla} f^{\nabla} \log h + \left(e^{\frac{f^{\nabla}}{f^{\nabla\nabla}} \cdot h^{f^{\nabla\nabla}}}\right)^{\nabla} - \left(e^{\frac{f^{\nabla}}{f^{\nabla\nabla}}} \cdot h^{\frac{f^{\nabla}}{f^{\nabla\nabla}} \cdot f^{\nabla\nabla}}\right)^{\nabla} + c_3^{\nabla}$$

but

$$\left(e^{\frac{f^{\nabla}}{f^{\nabla\nabla}} \cdot h^{f^{\nabla\nabla}}}\right)^{\nabla} = f^{\nabla} h^{f^{\nabla\nabla}} - \frac{c}{f^{\nabla\nabla}} \cdot f^{\nabla} h^{f^{\nabla\nabla}} + c \cdot \log h \cdot f^{\nabla} h^{f^{\nabla\nabla}}$$

and

$$\left(e^{\frac{f^{\nabla}}{f^{\nabla\nabla}}} \cdot h^{f^{\nabla}}\right)^{\nabla} = f^{\nabla} - \frac{c \cdot f^{\nabla}}{f^{\nabla\nabla}} + f^{\nabla\nabla} \cdot f^{\nabla} \log h$$

from (2) and (3), we conclude,

$$f^{\nabla}(x) \cdot h^{f^{\nabla\nabla}(x)} \cdot \left(\frac{e^{h^{c}}}{e \cdot h^{c}}\right)^{\frac{f^{\nabla\nabla}}{c}} = f^{\nabla} + f^{\nabla\nabla} f^{\nabla} \log h + \left(e^{\frac{f^{\nabla}}{f^{\nabla\nabla}} \cdot h^{f^{\nabla\nabla}}}\right)^{\nabla} - \left(e^{\frac{f^{\nabla}}{f^{\nabla\nabla}}} \cdot h^{\frac{f^{\nabla}}{f^{\nabla\nabla}} \cdot f^{\nabla\nabla}}\right)^{\nabla} + c_3^{\nabla}$$

$$= f^{\nabla} + f^{\nabla\nabla} f^{\nabla} \log h$$

$$+ f^{\nabla} h^{f^{\nabla\nabla}} - \frac{c}{f^{\nabla\nabla}} \cdot f^{\nabla} h^{f^{\nabla\nabla}} + c \cdot \log h \cdot f^{\nabla} h^{f^{\nabla\nabla}}$$

$$- f^{\nabla} + \frac{c \cdot f^{\nabla}}{f^{\nabla\nabla}} - f^{\nabla\nabla} \cdot f^{\nabla} \log h + c_3^{\nabla}$$

so that,

$$c_3^{\nabla} = f^{\nabla} \cdot h^{f^{\nabla\nabla}} \cdot \left(\frac{e^{h^{c}}}{e \cdot h^{c}}\right)^{\frac{f^{\nabla\nabla}}{c}} - f^{\nabla} h^{f^{\nabla\nabla}} \left(1 - \frac{c}{f^{\nabla\nabla}} + c \cdot \log h\right) - \frac{c \cdot f^{\nabla}}{f^{\nabla\nabla}}$$

Example 1: Let f be such that $f^{\nabla\nabla\nabla} = 1$. Then

$$\begin{aligned} f^{\nabla\nabla\nabla} &= 1 \\ f^{\nabla\nabla} &= x \\ f^{\nabla} &= e^x \\ f &= e^{Ei(x)} \end{aligned}$$

Replacing these values in

$$c_3^{\nabla} = f^{\nabla} \cdot h^{f^{\nabla\nabla}} \cdot \left(\frac{e^{h^c}}{e \cdot h^c}\right)^{\frac{f^{\nabla\nabla}}{c}} - f^{\nabla} h^{f^{\nabla\nabla}}\left(1 - \frac{c}{f^{\nabla\nabla}} + c \cdot \log h\right) - \frac{c \cdot f^{\nabla}}{f^{\nabla\nabla}}$$

we get

$$\begin{aligned} c_3^{\nabla} &= e^x \cdot h^x \cdot \left(\frac{e^h}{e \cdot h}\right)^x - e^x \cdot h^x\left(1 - \frac{1}{x} + \log h\right) - \frac{e^x}{x} \\ &= e^{h \cdot x} - e^x \cdot h^x\left(1 - \frac{1}{x} + \log h\right) - \frac{e^x}{x} \end{aligned}$$

so that

$$\begin{aligned} \log c_3 &= \int \frac{e^{xh}}{x} dx - \int \frac{(e \cdot h)^x}{x}\left(1 - \frac{1}{x} + \log h\right) dx - \int \frac{e^x}{x^2} dx \\ &= Ei(xh) - \frac{(e \cdot h)^x}{x} - Ei(x) + \frac{e^x}{x} + k \end{aligned}$$

Substituting however back in (1) we get an identity

$$\begin{aligned} f(x \cdot h) &= f(x) \cdot h^{f^{\nabla}(x)} \cdot \left(\frac{e^{h^{f^{\nabla\nabla}}}}{e \cdot h^{f^{\nabla\nabla}}}\right)^{\frac{f^{\nabla}}{f^{\nabla\nabla}}} \cdot c_3 \\ e^{Ei(xh)} &= e^{Ei(x)} \cdot h^{e^x}\left(\frac{e^{h^x}}{e \cdot h^x}\right)^{\frac{e^x}{x}} \cdot \frac{e^{Ei(xh)} e^{\frac{e^x}{x}}}{e^{Ei(x)} \cdot e^{\frac{(eh)^x}{x}}} \end{aligned}$$